Tree Search Configuration: Cutting Planes and Beyond

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Based on joint work with:
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Sample Complexity of Tree Search Configuration: Cutting Planes and Beyond. *NeurIPS'21 Spotlight*
Improved Sample Complexity Bounds for Branch-and-Cut. *CP'22*
Structural Analysis of Branch-and-Cut and the Learnability of Gomory Mixed Integer Cuts. *NeurIPS'22*

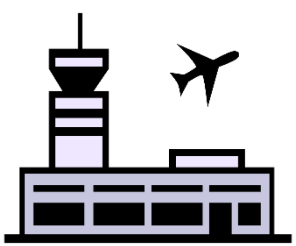
INFORMS 2022

Integer programming

- Integer program (IP) in standard form:

$$\begin{aligned} &\text{Max } \boldsymbol{c} \cdot \boldsymbol{x} \\ &\text{s.t. } A\boldsymbol{x} \leq \boldsymbol{b} \\ &\qquad \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$

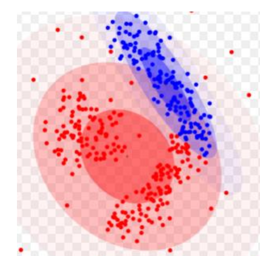
- One of the most useful and widely applicable optimization techniques



Scheduling

Routing

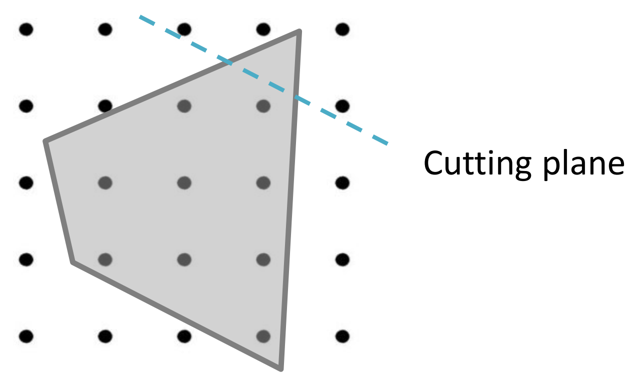
Combinatorial auctions



Clustering

Summary of contributions

- *Cutting planes*: responsible for breakthrough speedups of IP solvers in last three decades
  - Many ways to configure how IP solvers (e.g. CPLEX, Gurobi) choose cutting planes



- Our contribution: first formal theory for using machine learning to select cutting planes

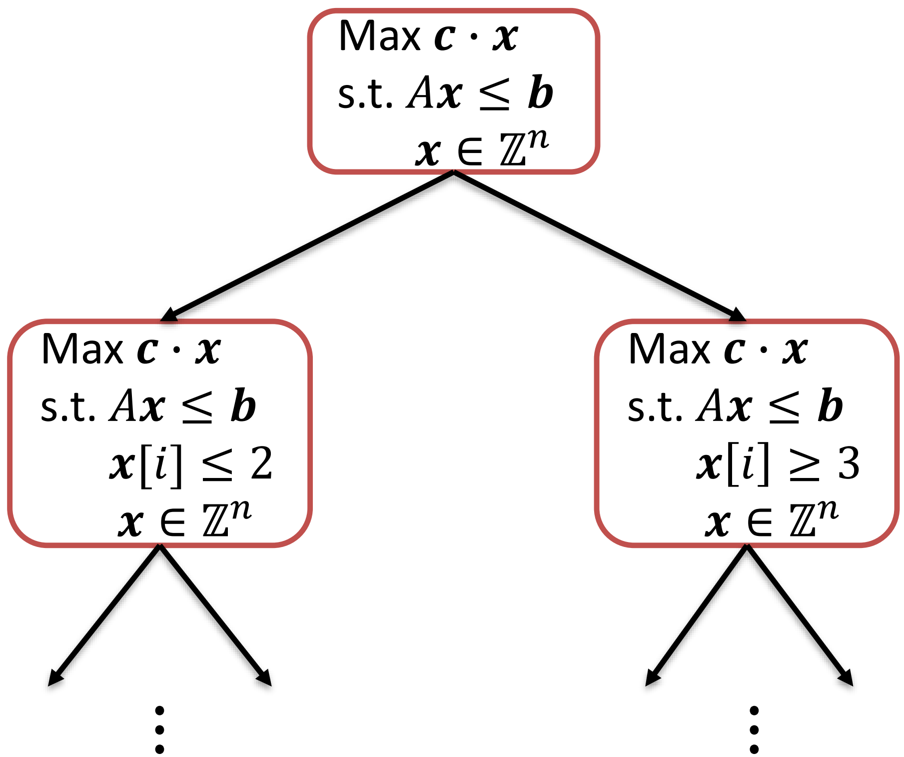
Branch-and-bound

- Powerful tree-search algorithm used to solve IPs in practice
- Uses the linear programming (LP) relaxation to do an informed search through the set of feasible integer solutions

IP		LP relaxation
Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \in \mathbb{Z}^n$	$\longrightarrow$	Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \in \mathbb{R}^n$

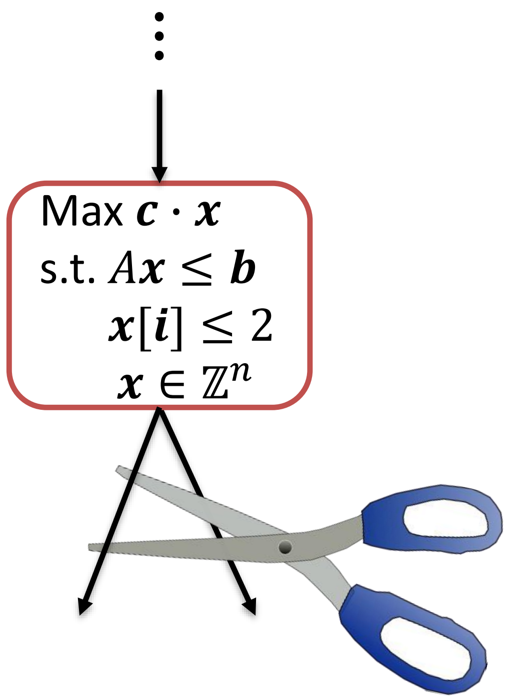
Branch-and-bound: branching

- Choose variable *i* to branch on.
- Generate one subproblem with $\boldsymbol{x}[i] \leq \lfloor \boldsymbol{x}_{\mathrm{LP}}^{*}[i] \rfloor$ another with $\boldsymbol{x}[i] \geq \lceil \boldsymbol{x}_{\mathrm{LP}}^{*}[i] \rceil$



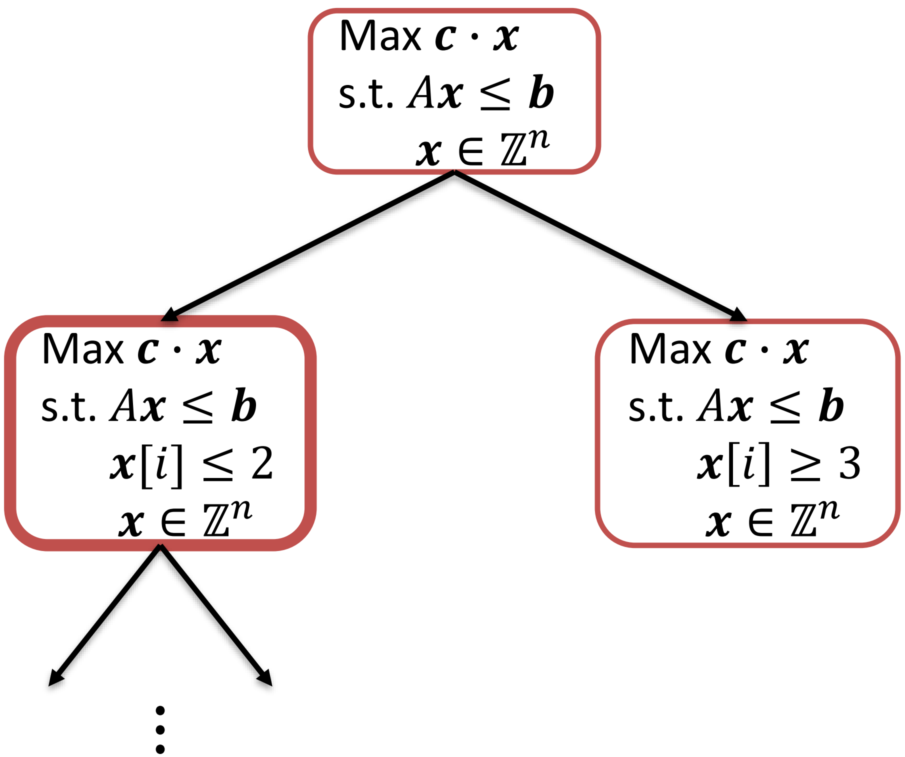
Branch-and-bound: pruning

- Prune subtrees if
  - LP relaxation at a node is integral, infeasible, or
  - (Bounding) LP optimal *worse* than best feasible integer solution found so far



Branch-and-bound: node selection

- At every stage, need to choose a leaf to explore further
- Variety of heuristics (e.g. *best-bound-first* chooses the node with the smallest LP objective)



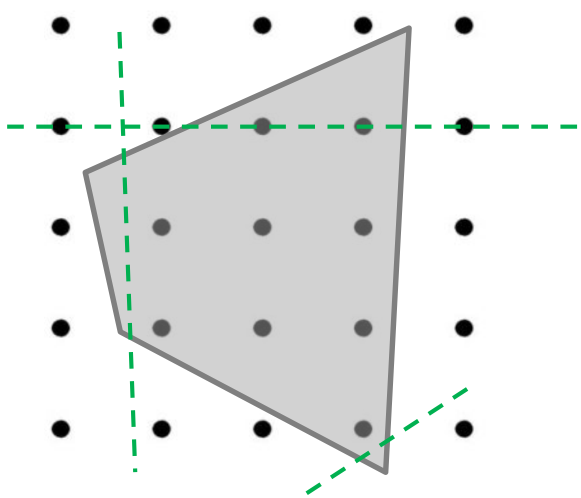
Branch-and-cut

- Branch-and-bound, but at each node may add *cutting planes*

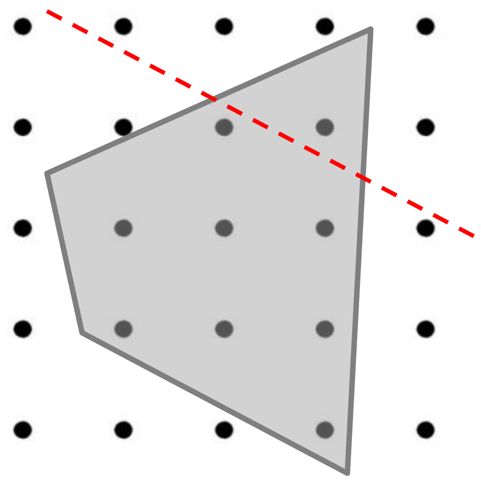
- Method of getting tighter LP relaxation bounds, and thus pruning subtrees sooner

Cutting planes

- Constraint $\boldsymbol{\alpha x} \leq \beta$ is a *valid cutting plane* if it does not cut off any integer feasible points



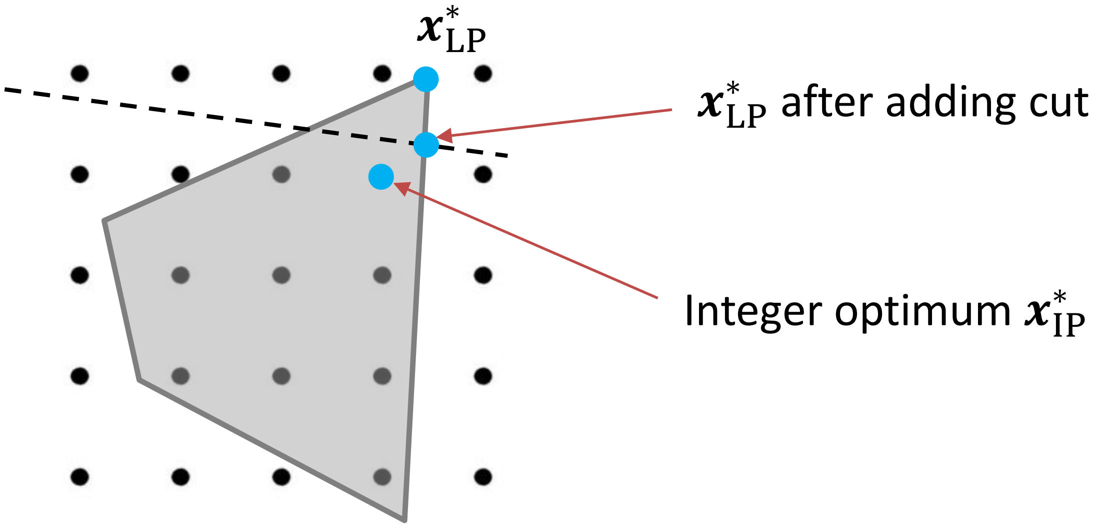
Valid cutting planes



An invalid cutting plane

Cutting planes

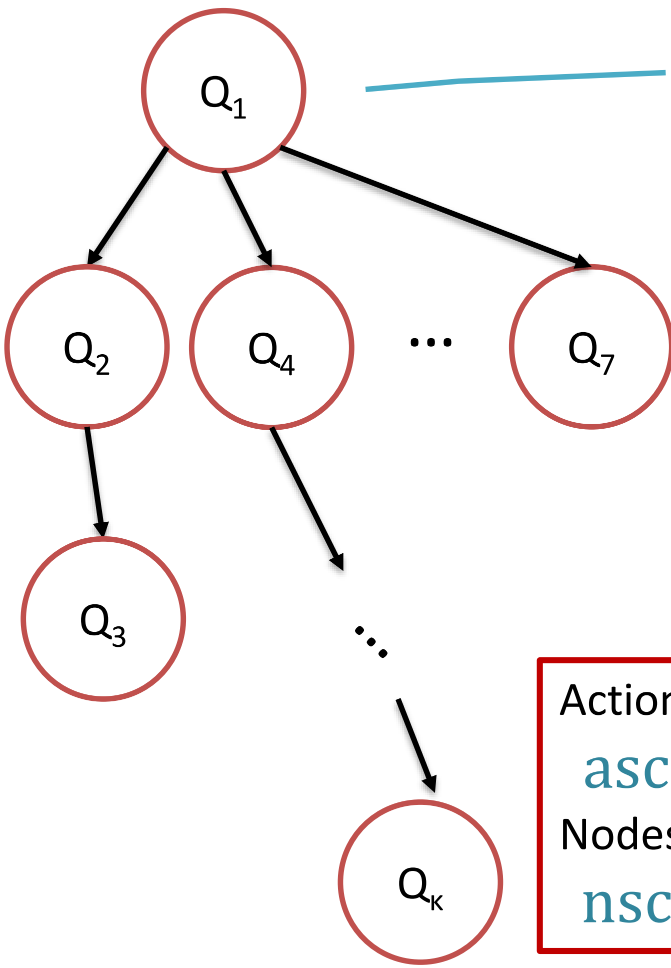
- If $\boldsymbol{\alpha x} \leq \beta$ is valid and separates the LP optimum, can speed up B&C by pruning nodes sooner



Tuning branch-and-cut

- Solvers like CPLEX, Gurobi have *numerous* parameters that control various aspects of the search (CPLEX has 170 page manual describing 172 parameters)

Parameterized tree search



- Select node Q that maximizes node selection rule nscore(T, Q)
  - Select action A that maximizes action score ascore(T, Q, A)
  - Either prune tree at Q, or add children
  - Continue until all nodes are pruned

Actions chosen using mixture of scoring rules:

$$\text{ascore} = \boldsymbol{\mu} \cdot \text{ascore}_1 + (1 - \boldsymbol{\mu}) \cdot \text{ascore}_2$$

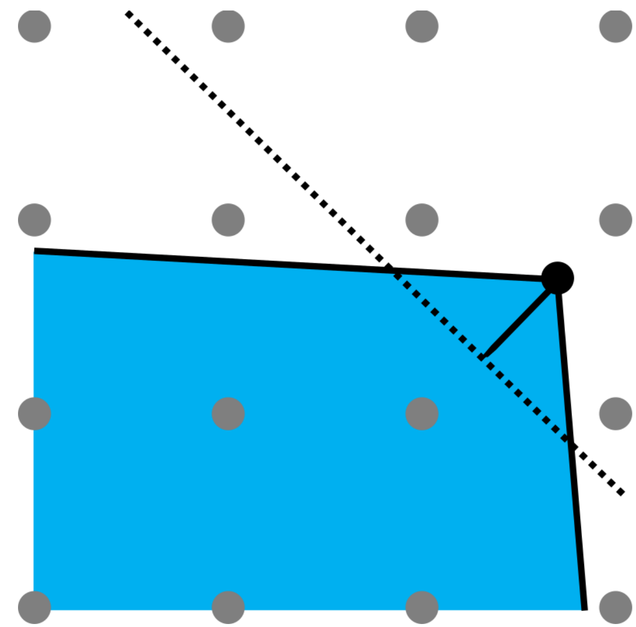
Nodes chosen using mixture of scoring rules:

$$\text{nscore} = \boldsymbol{\lambda} \cdot \text{nscore}_1 + (1 - \boldsymbol{\lambda}) \cdot \text{nscore}_2$$

Example of a scoring rule: efficacy

<u>*Efficacy:*</u>

distance between cut and $\boldsymbol{x}^*_{\mathrm{LP}}$

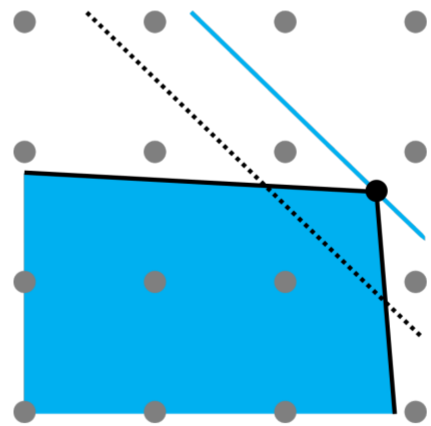


$$\mathrm{score}_1(\boldsymbol{\alpha}^T\boldsymbol{x} \leq \beta) = \frac{\boldsymbol{\alpha}\boldsymbol{x}^*_{\mathrm{LP}} - \beta}{\|\boldsymbol{\alpha}\|_2}$$

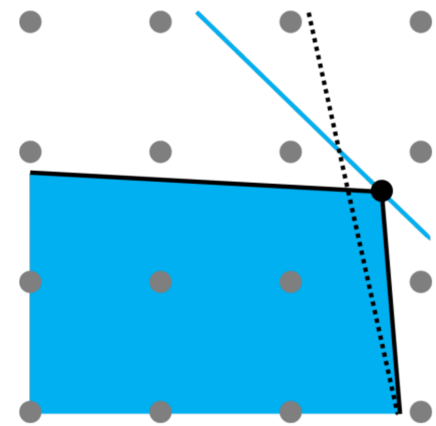
Example of a scoring rule: parallelism

Parallelism:

angle between cut and objective



Better parallelism



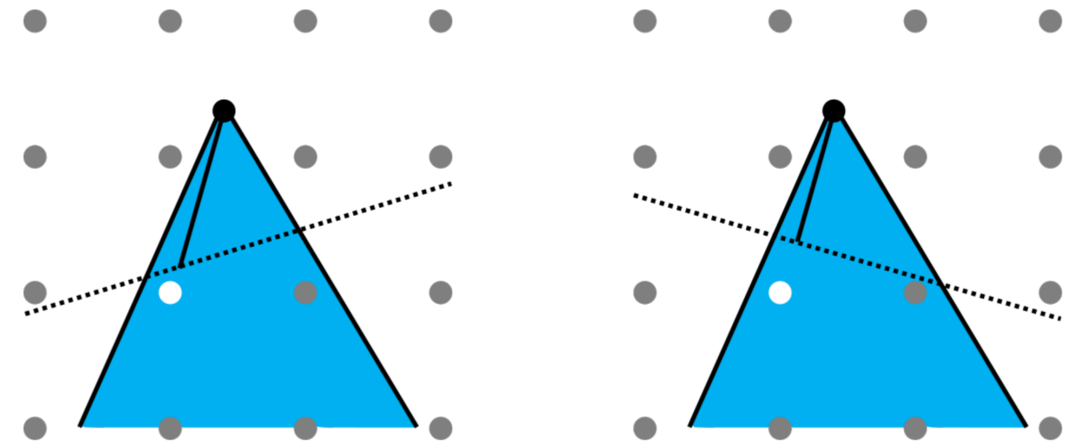
Worse parallelism

$$\text{score}_2(\boldsymbol{\alpha}^T \boldsymbol{x} \leq \beta) = \frac{|\boldsymbol{c}\boldsymbol{\alpha}|}{\|\boldsymbol{\alpha}\|_2 \|\boldsymbol{c}\|_2}$$

Example of a scoring rule: directed cutoff

<u>Directed cutoff:</u>

distance between cut and $\boldsymbol{x}^*_{\mathrm{LP}}$, in direction of current best integer solution



Better directed cutoff   Worse directed cutoff

$$\mathrm{score}_3(\boldsymbol{\alpha}^T\boldsymbol{x} \leq \beta) = \frac{\boldsymbol{\alpha}\boldsymbol{x}^*_{\mathrm{LP}} - \beta}{|\boldsymbol{\alpha}(\overline{\boldsymbol{x}} - \boldsymbol{x}^*_{\mathrm{LP}})|} \cdot \|\overline{\boldsymbol{x}} - \boldsymbol{x}^*_{\mathrm{LP}}\|_2$$

Scoring rules

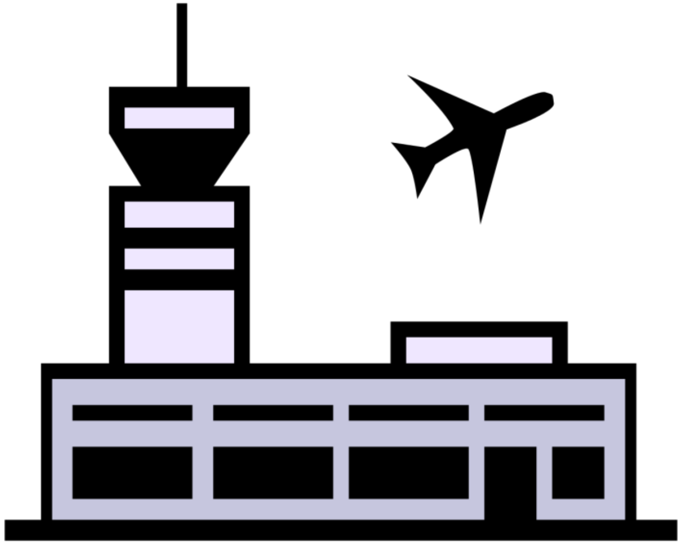
- Open source solver SCIP uses hard-coded mixture of scores to choose cuts
  $\frac{3}{5}\mathrm{score}_1 + \frac{1}{10}\mathrm{score}_2 + \frac{1}{2}\mathrm{score}_3 + \frac{1}{10}\mathrm{score}_4$

Generalization guarantees for tree search and branch-and-cut

Distribution-dependent parameter selection of μ, λ

Learning to tune tree search

Best parameters for
airline-scheduling IPs…



…might not be useful for
combinatorial-auction IPs
solved by a sourcing firm

Learning to tune branch-and-cut

If a certain set of parameters yields small average branch-and-cut tree size over IP samples...

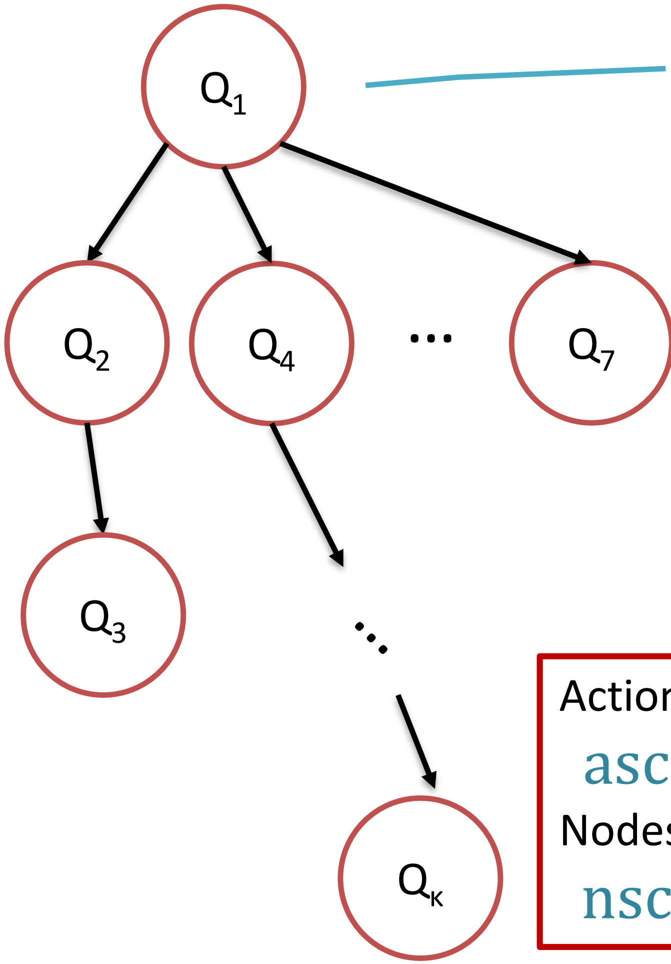
$$\begin{array}{l}\text{Max } \boldsymbol{c_1} \cdot \boldsymbol{x} \\ \text{s.t. } A_1\boldsymbol{x} \leq \boldsymbol{b_1} \\ \quad\quad \boldsymbol{x} \in \mathbb{Z}^n\end{array} \quad \bullet\bullet\bullet \quad \begin{array}{l}\text{Max } \boldsymbol{c}_N \cdot \boldsymbol{x} \\ \text{s.t. } A_N\boldsymbol{x} \leq \boldsymbol{b_N} \\ \quad\quad \boldsymbol{x} \in \mathbb{Z}^n\end{array} \quad \sim D$$

IP 1 IP N

...is it likely to yield a small branch-and-cut tree on a fresh IP?

$$\begin{array}{l}\text{Max } \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t. } A\boldsymbol{x} \leq \boldsymbol{b} \\ \quad\quad \boldsymbol{x} \in \mathbb{Z}^n\end{array} \quad \sim D$$

Parameterized tree search



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Nodes chosen using mixture of scoring rules:

$$\text{nscore} = \boldsymbol{\lambda} \cdot \text{nscore}_1 + (1 - \boldsymbol{\lambda}) \cdot \text{nscore}_2$$

Generalization guarantee for tree search

Theorem [BPSV CP'22]: For all μ, λ, difference between average training performance and expected performance when μ, λ is used to select actions and nodes throughout the tree is (whp)

$$\tilde{O}\left(H\sqrt{\frac{\Delta^2 \log k + \Delta \log b}{N}}\right)$$

Holds for any (unknown) distribution over tree-search problem instances

Δ = tree depth
k = tree branching factor
b = # actions available at each node
H = cap on size of tree

First guarantee that handles multiple critical aspects of branch-and-cut:
Node selection, branching, and cutting plane selection

Sample complexity of tuning tree search

Theorem [BPSV CP'22]: For all μ, λ, the number of samples so that the difference between average training performance and expected performance when μ, λ is used to select actions and nodes throughout the tree is (whp) at most ε is

$$\tilde{O}\left(\frac{H^2}{\varepsilon^2}\left(\Delta^2 \log k + \Delta \log b\right)\right)$$

Δ = tree depth
k = tree branching factor
b = # actions available at each node
H = cap on size of tree

First guarantee that handles multiple critical aspects of branch-and-cut:
Node selection, branching, and cutting plane selection

Back to branch-and-cut

- Our result implies polynomial bounds for:
  - Branching: single-variable, multi-variable, branching on general disjunctions with bounded coefficients,…
  - Cutting planes: cover cuts, clique cuts, any cuts derived from simplex tableau (Chvátal cuts, Gomory mixed integer cuts)
  - Allows node selection to be tuned simultaneously
- Prior work
  - [Balcan et al. ICML'18] studied single-variable branching with pathwise scoring rules (our result recovers theirs)

Knapsack cover cuts – an experiment

- Set of items N, item $i \in N$ has value $p_i \geq 0$ and weight $w_i \geq 0$
- Set of knapsacks K, knapsack $k \in K$ has capacity $W_k \geq 0$
- *Goal:* find feasible packing of maximum weight

$$\begin{array}{lll} \text{maximize} & \Sigma_{i \in N} \Sigma_{k \in K} p_i x_{k,i} & \\ \text{subject to} & \Sigma_{i \in N} w_i x_{k,i} \leq W_k & \forall k \in K \\ & \Sigma_{k \in K} x_{k,i} \leq 1 & \forall i \in N \\ & x_{k,i} \in \{0,1\} & \forall i \in N, k \in K \end{array}$$

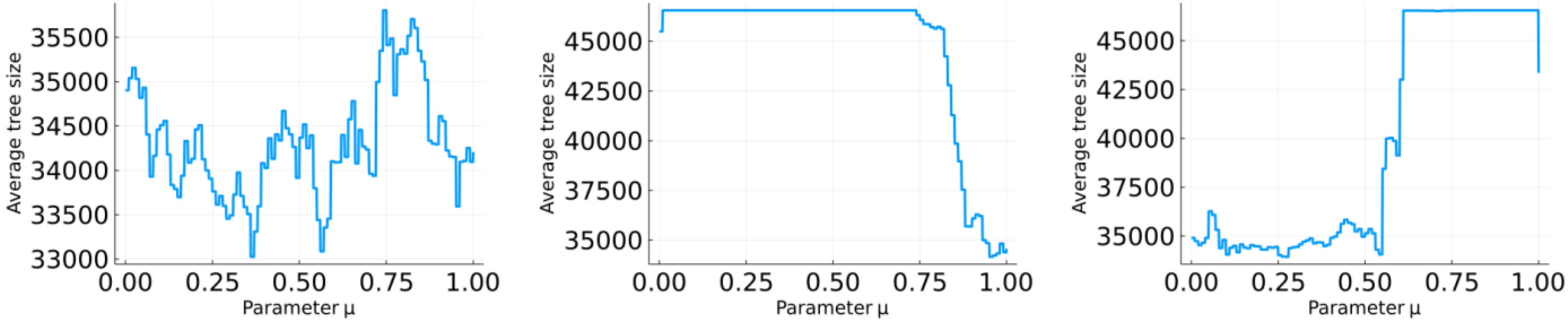
Knapsack cover cuts – an experiment

- Cover cut for knapsack k: if $w_1 + w_2 + w_3 \geq W_k$ (items 1, 2, 3 are jointly too heavy for knapsack k), can enforce the constraint $x_{k,1} + x_{k,2} + x_{k,3} \leq 2$

- We tune convex combinations of cut scoring rules to control the addition of cover cuts* throughout the branch-and-cut tree

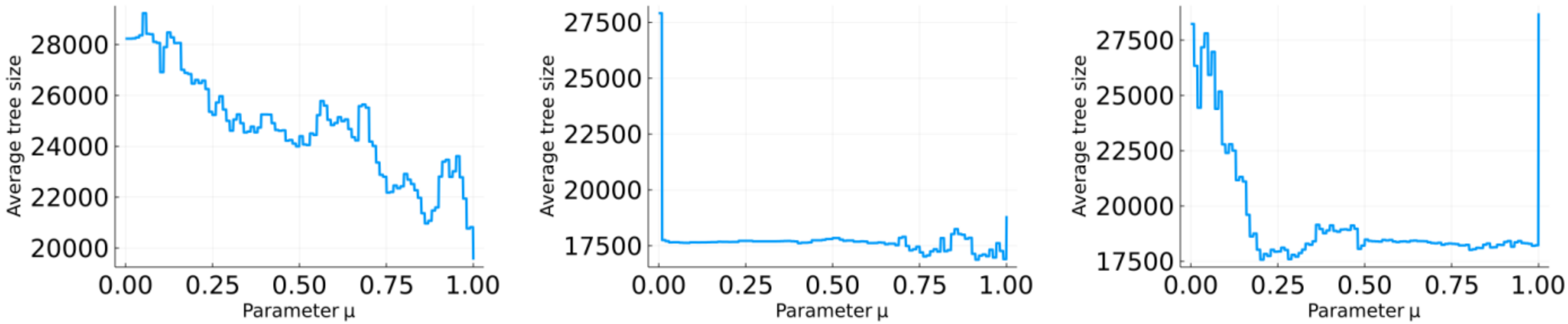
*actually a special kind of cover cut: *extended minimal cover cuts*

Knapsack cover cuts – an experiment



(a) $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{P}$. **(b)** $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{D}$. **(c)** $\mu \cdot \mathtt{D} + (1 - \mu) \cdot \mathtt{P}$.

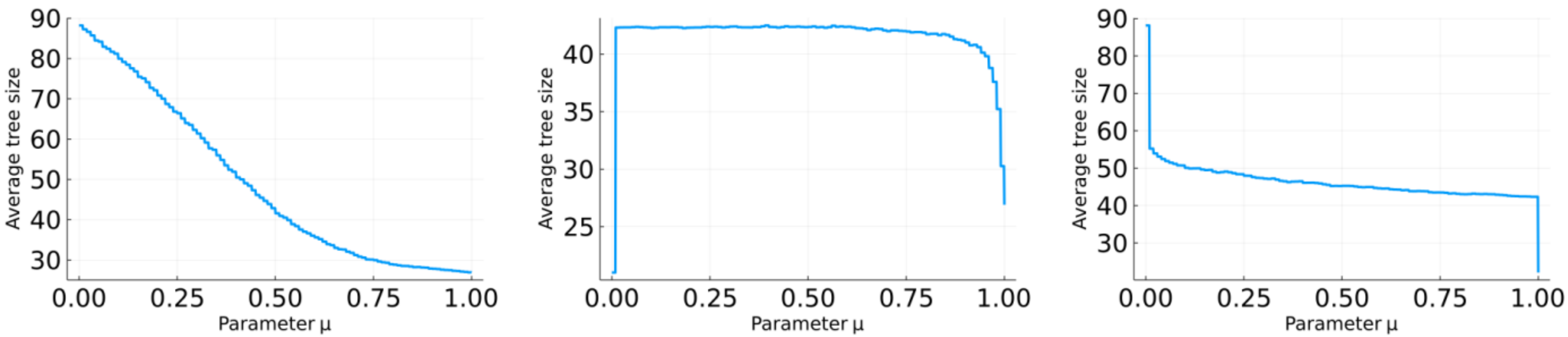
Figure 1 Chvátal distribution with 35 items and 2 knapsacks.



(a) $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{P}$. **(b)** $\mu \cdot \mathtt{E} + (1 - \mu) \cdot \mathtt{D}$. **(c)** $\mu \cdot \mathtt{D} + (1 - \mu) \cdot \mathtt{P}$.

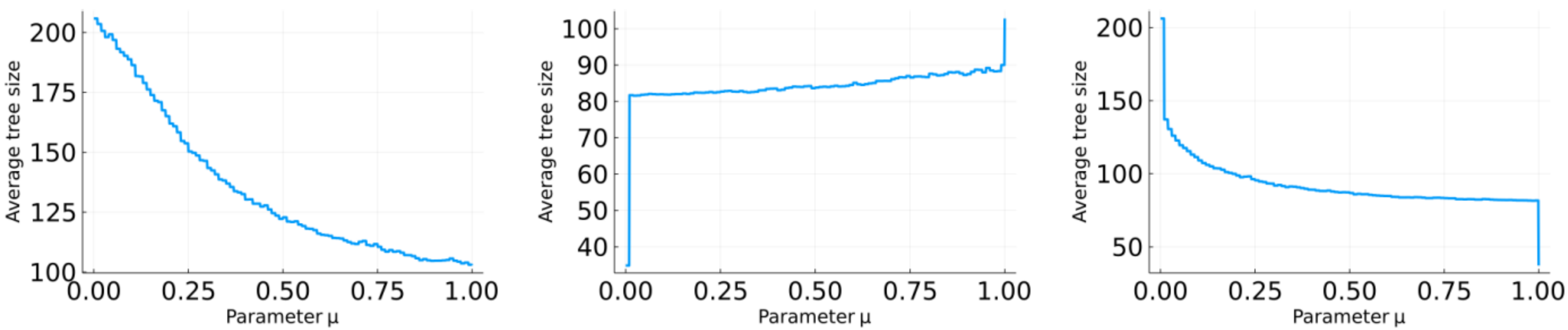
Figure 2 Chvátal distribution with 35 items and 3 knapsacks.

Knapsack cover cuts – an experiment



(a) $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{P}$. **(b)** $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{D}$. **(c)** $\mu \cdot \mathtt{D} + (1-\mu) \cdot \mathtt{P}$.

Figure 3 Reverse Chvátal distribution with 100 items and 10 knapsacks.



(a) $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{P}$. **(b)** $\mu \cdot \mathtt{E} + (1-\mu) \cdot \mathtt{D}$. **(c)** $\mu \cdot \mathtt{D} + (1-\mu) \cdot \mathtt{P}$.

Figure 4 Reverse Chvátal distribution with 100 items and 15 knapsacks.

Structure of branch-and-cut

- [BPSV NeurIPS'21]: Analysis of Chvátal-Gomory cuts and policies for adding them throughout the B&C tree
- [BPSV NeurIPS'22]: Analysis of Gomory Mixed Integer cuts
  - Requires a deeper mathematical analysis of the geometric and combinatorial structure of B&C